Bayesian inference for reliability of systems and networks using the survival signature

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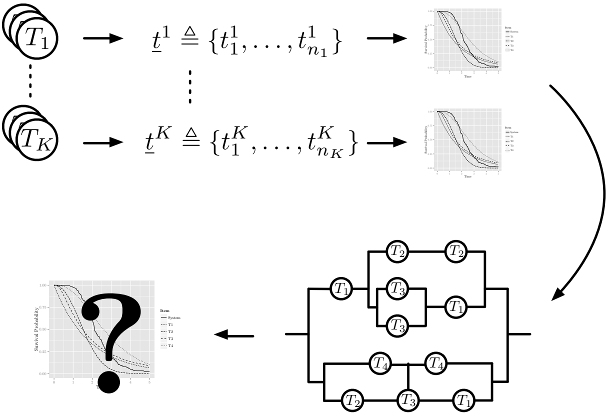
Problem setting

Test data available on a components to be used in a system.

Objective: inference on system/network reliability given component test data.



A nonparametric model of components

At a fixed time t, probability component of type k functions is Bernoulli(p_t^k) for some unknown p_t^k.

$\implies$ number functioning at time t in iid batch of n_k is Binomial(n_k, p_t^k).

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Let $S_t^k \in \{0, 1, \dots, n_k\}$ be number of working components in test batch of n_k components of type k. Then,

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Given test data $\underline{t}^k = \{t_1^k, \dots, t_{n_k}^k\}$, for each t we can form corresponding observation from Binomial model

$$s_t^k \triangleq \sum_{i=1}^{n_k} \mathbb{I}(t_i^k > t)$$

Bayesian inference for nonparametric model

Taking prior $p_t^k \sim \text{Beta}(\alpha_t^k, \beta_t^k)$, exploit conjugacy result

$$p_t^k \mid s_t^k \sim \text{Beta}(\alpha_t^k + s_t^k, \beta_t^k + n_k - s_t^k)$$

Then, posterior predicitive for number of components surviving in a new batch of m_k components is

$$C_t^k \mid s_t^k \sim \text{Beta-binomial}(m_k, \alpha_t^k + s_t^k, \beta_t^k + n_k - s_t^k)$$

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Summary: for any fixed t, s_t^k provides a minimal sufficient statistic for computing posterior predictive distribution of the number of components surviving to t in a new batch, without any parametric model for component lifetime being assumed.

Propagating uncertainty to the system

Now take collection of component types $k \in \{1, \ldots, K\}$, each with test data $\underline{t} = \{\underline{t}^1, \ldots, \underline{t}^k\}$, and corresponding collection of minimal sufficient statistics for a fixed t, $\{s_t^1, \ldots s_t^K\}$.

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Survival probability for a new system S^* comprising these component types follows naturally via posterior predictive and surival signature:

$$P(T_{S^*} > t \,|\, s_t^1, \ldots s_t^K)$$

$$= \int \cdots \int P(T_{S^*} > t \,|\, p_t^1, \ldots p_t^K) P(p_t^1 \mid s_t^1) \ldots P(p_t^K \mid s_t^K) \, dp_t^1 \ldots dp_t^K$$

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&= \int \cdots \int P(T_{S^*} > t \,|\, p_t^1, \dots p_t^K) P(p_t^1 \,|\, s_t^1) \dots P(p_t^K \,|\, s_t^K)\, dp_t^1 \dots dp_t^K \\
&= \int \cdots \int \left[\sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \dots, l_K) P\left(\bigcap_{k=1}^{K} \{ C_t^k = l_k \,|\, p_t^k \} \right) \right] \\
&\qquad \times P(p_t^1 \,|\, s_t^1) \dots P(p_t^K \,|\, s_t^K)\, dp_t^1 \dots dp_t^K
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$$P(\,T_{S^*} > t \,|\, s_t^1, \dots s_t^K)$$

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$$= \int \cdots \int \left[\sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \dots, l_K) P\left(\bigcap_{k=1}^{K} \{\, C_t^k = l_k \,|\, p_t^k \} \right) \right] \times P(p_t^1 \mid s_t^1) \dots P(p_t^K \mid s_t^K)\, dp_t^1 \dots dp_t^K$$

$$= \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \dots, l_K) \prod_{k=1}^{K} \int P(C_t^k = l_k \,|\, p_t^k) P(p_t^k \mid s_t^k)\, dp_t^k$$

Final integral is simply the posterior predictive (Beta-binomial).

System survival probability

$$P(T_{S^*} > t \mid s_t^1, \dots s_t^K)$$
$$= \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \dots, l_K)$$
$$\times \prod_{k=1}^{K} \binom{m_k}{l_k} \frac{\mathrm{B}(l_k + \alpha_t^k + s_t^k, m_k - l_k + \beta_t^k + n_k - s_t^k)}{\mathrm{B}(\alpha_t^k + s_t^k, \beta_t^k + n_k - s_t^k)}$$

Incredibly easy to implement this algorithmically since survival signature has factorised the survival function by component type!

Why not structure function?

$$\phi(\underline{x}) = \prod_{j=1}^{s} \left(1 - \prod_{i \in C_j} (1 - x_i) \right)$$

where $\{C_1, \ldots, C_s\}$ is the collection of minimal cut sets of the system. Recall don't need $x \in \{0, 1\}$ — we can plug in probabilities. So why not?

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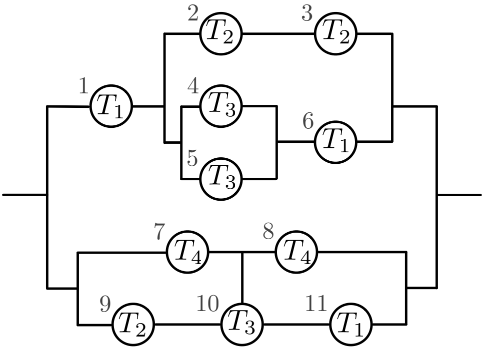
$$P(\, T_{S^*} > t \,|\, s_t^1, \ldots s_t^K) = \int \cdots \int \phi(p_t^{x_1}, \ldots, p_t^{x_n}) P(p_t^1 \mid s_t^1) \ldots P(p_t^K \mid s_t^K) \, dp_t^1 \ldots dp_t^K$$

where $p_t^{x_i}$ is the element of $\{p_t^1, \ldots, p_t^K\}$ corresponding to component i.

Have fun with that integral for large $K \ldots$!

Example system layout, $K = 4$, $n = 11$

Example system:



$T_1 \sim \text{Exp}(\lambda_1 = 0.55)$

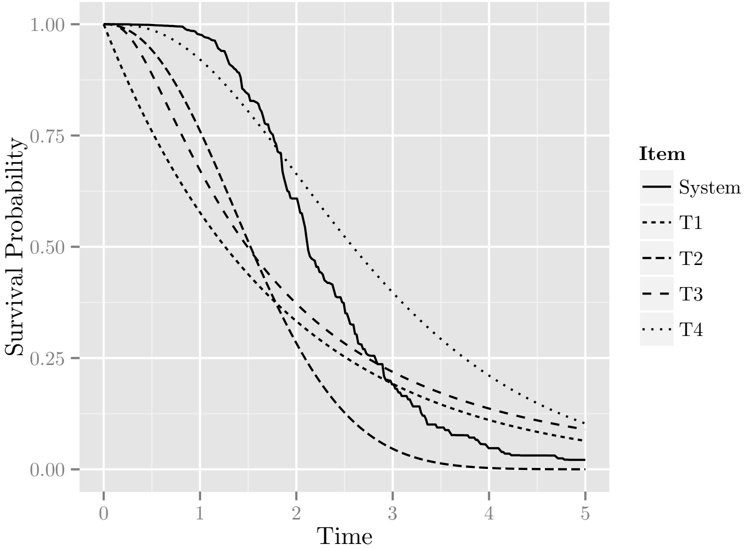
$T_2 \sim \text{Wei}(\lambda_2 = 1.8, \gamma_1 = 2.2)$

$T_3 \sim \text{Log-N}(\mu = 0.4, \tau = 1.234)$

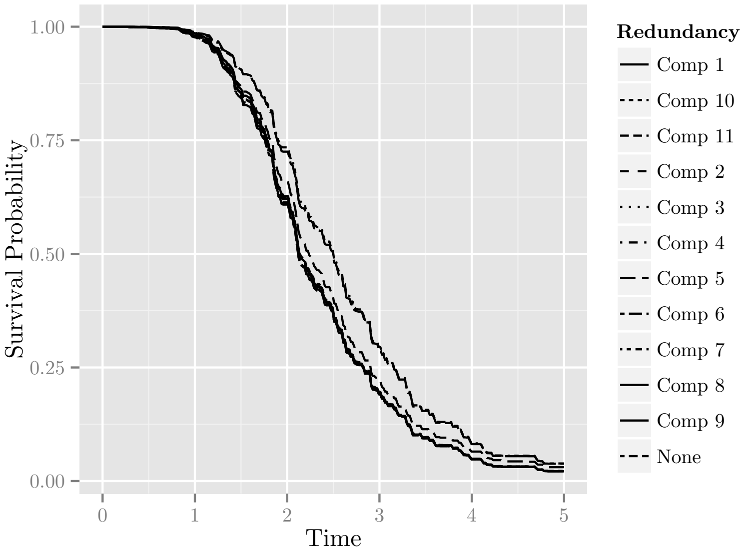
$T_4 \sim \text{Gam}(\lambda_3 = 0.9, \gamma_2 = 3.2)$

Simulated test data with $n_k = 100 \; \forall \, k$

Posterior predictive survival curves



Optimal redundancy?



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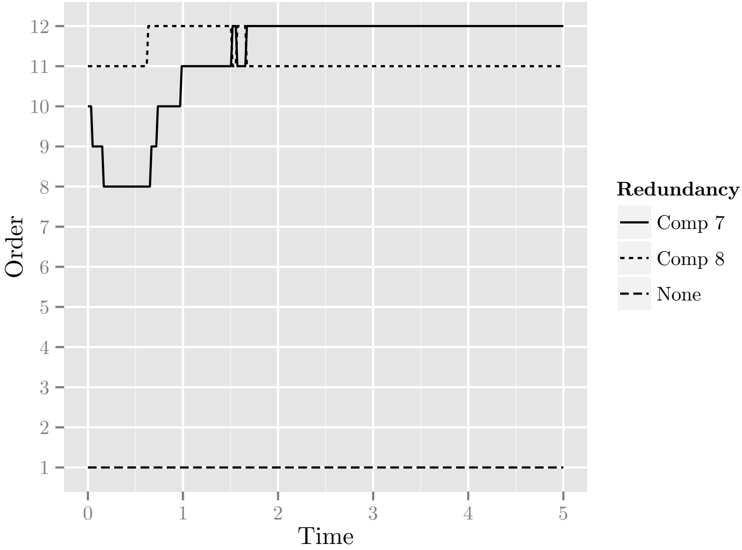

Parametric models of components

The survival signature achieves the same factorisation of system lifetime when using parametric models for the components.

Model the lifetime of component k directly via likelihood function f_k

$$T_k \sim f_k(t; \psi_k)$$

As before, given test data $\underline{t}^k = \{t_1^k, \dots, t_{n_k}^k\}$ for component k, posterior density is:

$$f_{\Psi_k \,|\, \underline{T}^k}(\psi_k \,|\, \underline{t}^k) \propto f_{\Psi_k}(\psi_k) \prod_{i=1}^{n_k} f_k(t_i^k; \psi_k)$$

$$P(T_{S^*} > t \,|\, \underline{t}^1, \dots \underline{t}^K)$$
$$= \int \cdots \int P(T_{S^*} > t \,|\, \psi_1, \dots \psi_K) f_{\Psi_1 \,|\, \underline{T}^1}(d\psi_1 \,|\, \underline{t}^1) \dots f_{\Psi_K \,|\, \underline{T}^K}(d\psi_K \,|\, \underline{t}^K)$$

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\end{aligned}
$$

Final term posterior predictive of l_k components of type k surviving to t.

Computing the integral for arbitrary models

Three possibilities. The posterior, $f_{\Psi_k \mid \underline{T}^k}(d\psi_k \mid \underline{t}^k)$, is:

1. in closed form and integral tractable;
2. known distribution, but the integral is intractable;
3. not in closed form.

Also, note the integral is just:

$$\mathbb{E}_{\Psi_k \mid \underline{T}^k}\left[[F_k(t;\psi_k)]^{m_k - l_k} [1 - F_k(t;\psi_k)]^{l_k} \right]$$

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Thus, for samples $\psi_k^{(1)}, \dots, \psi_k^{(N)} \sim \Psi_k \,|\, \underline{T}^k$ we can always fall back to evaluating:

$$\frac{1}{N}\sum_{i=1}^{N}[F_k(t;\psi_k^{(i)})]^{m_k - l_k}[1 - F_k(t;\psi_k^{(i)})]^{l_k}$$

$$\xrightarrow{N\to\infty} \mathbb{E}_{\Psi_k \,|\, \underline{T}^k}\left[[F_k(t;\psi_k)]^{m_k - l_k}[1 - F_k(t;\psi_k)]^{l_k} \right]$$

Posterior predictive survival curves for both methods