Data Science and Statistical Computing

Tutorial 5, Week 10 Solutions

Q1

(a)

$$\tilde{f}(x) = \frac{1}{2}\exp(-|x|) \quad \forall\, x \in \mathbb{R}$$

$$\begin{aligned}
\tilde{F}(x) &= \int_{-\infty}^{x} \tilde{f}(z)\,dz \\
&= \begin{cases} \frac{1}{2}\int_{-\infty}^{x} \exp(z)\,dz & \text{if } x \le 0 \\ \frac{1}{2} + \frac{1}{2}\int_{0}^{x} \exp(-z)\,dz & \text{if } x > 0 \end{cases}
\end{aligned}$$

The first integral is trivially e^x. For the second integral, you can either use a simple symmetry argument, or, to plough through the calculus directly we could use the substitution $u = -z \implies du = -dz$ and limits $u = -0 = 0$ to $u = -x$,

$$\int_0^x \exp(-z)\,dz = -\int_0^{-x} \exp(u)\,du = -e^u\Big|_0^{-x} = 1 - e^{-x}$$

Therefore,

$$\tilde{F}(x) = \begin{cases} \frac{e^x}{2} & \text{if } x \le 0 \\ 1 - \frac{e^{-x}}{2} & \text{if } x > 0 \end{cases}$$

The generalised inverse is thus,

$$\tilde{F}^{-1}(u) = \begin{cases} \log(2u) & \text{if } u \le \frac{1}{2} \\ -\log(2-2u) & \text{if } u > \frac{1}{2} \end{cases}$$

Hence, to inverse sample a standard Laplace random variable we would generate $U \sim \text{Unif}(0,1)$ and compute $\tilde{F}^{-1}(U)$ per above.

(b)

We require $c < \infty$ such that

$$\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) \le \frac{c}{2}\exp(-|x|) \quad \forall\, x \in \mathbb{R}$$

Both pdfs are symmetric about zero, so this is equivalent to:

$$\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) \le \frac{c}{2}\exp(-x) \quad \forall\, x \in [0,\infty)$$

In other words, we require:

$$c = \sup_{x \in [0,\infty)} \sqrt{\frac{2}{\pi}} \exp\left(-\frac{x^2}{2} + x\right)$$

As $x \to \infty$, the $-x^2/2$ dominates the $+x$, so the exponential term tends to zero and we can see there is no problem in the tail.

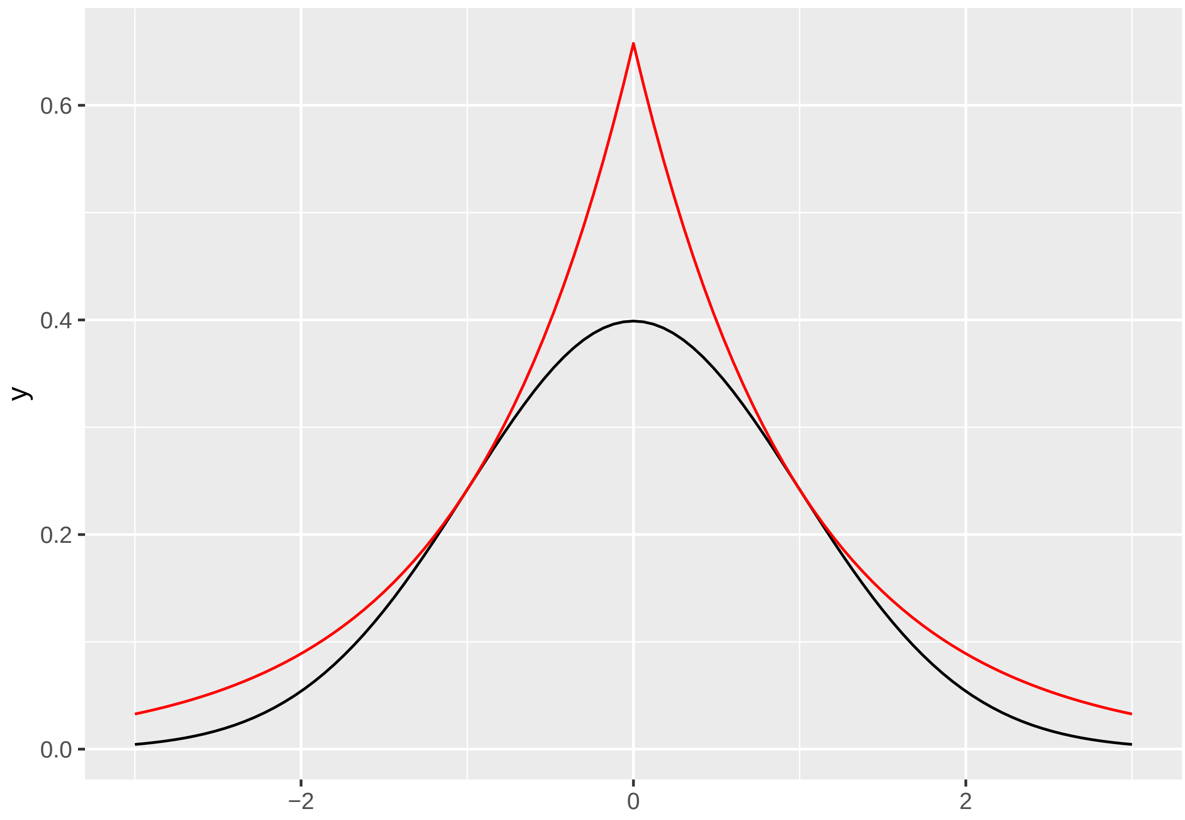
Now,

$$\frac{d}{dx} = \sqrt{\frac{2}{\pi}}(1-x)\exp\left(-\frac{x^2}{2} + x\right) \qquad \text{Simple chain rule application}$$
$$\frac{d^2}{dx^2} = \sqrt{\frac{2}{\pi}}x(x-2)\exp\left(-\frac{x^2}{2} + x\right) \qquad \text{Product and chain rules}$$

By inspection, $d/dx = 0 \iff x = 1$ and at this point $d^2/dx^2 < 0$ confirming this is a maximum. Hence,

$$c = \sqrt{\frac{2}{\pi}} \exp\left(\frac{1}{2}\right) \approx 1.3155$$

The following graph shows the standard Laplace distribution scaled by c in red, with the standard normal in black, confirming that c satisfies the inequality.



(c)

1. Set $a \leftarrow$ FALSE
2. While a is FALSE,

    i. Generate $U_1 \sim \text{Unif}(0,1)$

ii. Compute

$$X = \begin{cases} \log(2U_1) & \text{if } U_1 \leq \frac{1}{2} \\ -\log(2 - 2U_1) & \text{if } U_1 > \frac{1}{2} \end{cases}$$

which generates a proposal from the standard Laplace distribution.

iii. Generate $U_2 \sim \text{Unif}(0, 1)$

iv. If

$$U_2 \leq \frac{1}{1.3155}\sqrt{\frac{2}{\pi}} \exp\left(-\frac{X^2}{2} + |X|\right)$$

then set $a \leftarrow$ TRUE

3. Return X as a sample from $\text{N}(0, 1)$.

(d)

By Lemma 5.1, the expected number of iterations of the rejection sampler required to produce a sample is given by c, so since here $c \approx 1.3155 < 1.521$ we would favour using the standard Laplace distribution as a proposal. The Cauchy proposal would require $(1.521/1.3155 - 1)\,\% \approx 15.6\%$ more iterations to produce any given number of standard Normal simulations.

Q2

(a)

By kindergarden rules of probability,

$$\begin{aligned} \mathbb{P}(X \leq x \,|\, a \leq X \leq b) &= \frac{\mathbb{P}(a \leq X \leq b \cap X \leq x)}{\mathbb{P}(a \leq X \leq b)} \\ &= \frac{\mathbb{P}(a \leq X \leq b \cap X \leq x)}{F(b) - F(a)} \end{aligned}$$

Taking the numerator separately,

$$\begin{aligned} \mathbb{P}(a \leq X \leq b \cap X \leq x) &= \begin{cases} 0 & \text{if } x < a \\ \mathbb{P}(a \leq X \leq x) & \text{if } a \leq x \leq b \\ \mathbb{P}(a \leq X \leq b) & \text{if } x > b \end{cases} \\ &= \begin{cases} 0 & \text{if } x < a \\ F(x) - F(a) & \text{if } a \leq x \leq b \\ F(b) - F(a) & \text{if } x > b \end{cases} \end{aligned}$$

Therefore,

$$\mathbb{P}(X \leq x \,|\, a \leq X \leq b) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{F(x) - F(a)}{F(b) - F(a)} & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b \end{cases}$$

(b)

We simply want to find the generalised inverse of the cdf we just derived. So if $U \sim \text{Unif}(0, 1)$, then we solve for X in:

$$
\begin{aligned}
U &= \frac{F(X) - F(a)}{F(b) - F(a)} \\
\implies F(X) &= U\big(F(b) - F(a)\big) + F(a) \\
\implies X &= F^{-1}\Big(U\big(F(b) - F(a)\big) + F(a)\Big)
\end{aligned}
$$

(c)

For an Exponential distribution, $F(x) = 1 - e^{-\lambda x}$ and therefore $F^{-1}(u) = -\lambda^{-1}\log(1-u)$.

Truncating to $X \in [1, \infty) \implies a = 1, b = \infty$, thus $F(a) = 1 - e^{-\lambda}, F(b) = 1$ and so the inverse sampler is:

$$
\begin{aligned}
X &= F^{-1}\Big(U\big(F(b) - F(a)\big) + F(a)\Big) \\
&= F^{-1}\Big(U\big(1 - (1 - e^{-\lambda})\big) + (1 - e^{-\lambda})\Big) \\
&= F^{-1}\Big(e^{-\lambda}\big(U - 1\big) + 1\Big) \\
&= -\lambda^{-1}\log\Big(1 - \big(e^{-\lambda}\big(U - 1\big) + 1\big)\Big) \\
&= -\lambda^{-1}\log\Big(e^{-\lambda}\big(1 - U\big)\Big) \\
&= 1 - \lambda^{-1}\log\big(1 - U\big)
\end{aligned}
$$

(d)

Notice that the inverse sampler for the Exponential distribution truncated to $[1, \infty)$ is simply $1 + \big(-\lambda^{-1}\log(1-U)\big)$ ... in other words, inverse sample a standard Exponential and add 1!

This agrees with the well-known memoryless property of the Exponential distribution.

Note: obviously for other distributions it will not be so simple, this is a very special property of Exponential random variables.