Data Science and Statistical Computing

Assignment 4 Solutions

Q1

> The Exponential distribution has probability density function (pdf):
>
> $$\tilde{f}(y \mid \lambda) = \begin{cases} \lambda e^{-\lambda y} & \text{if } y \in [0, \infty) \\ 0 & \text{otherwise} \end{cases}$$
>
> where $\lambda > 0$.
>
> Simulate three values (pen-and-paper, not R) from this pdf via inverse transform sampling using the following values simulated from the Uniform$(0, 1)$ distribution:
>
> $$0.56, \quad 0.85, \quad 0.26$$

$$\begin{aligned} \tilde{F}(y) = \mathbb{P}(Y \le y) &= \int_0^y \lambda e^{-\lambda t} \, dt \\ &= -e^{-\lambda t}\Big|_{t=0}^{y} \quad \text{by substitution } (u = \lambda t) \text{ or inspection} \\ &= 1 - e^{-\lambda y} \end{aligned}$$

$$\implies \tilde{F}^{-1}(u) = -\lambda^{-1} \log(1 - u)$$

Therefore, we can use the three uniform simulations provided in the question to generate three Exponentially distributed simulations:

$$\begin{aligned} u = 0.56 &\implies y = -\lambda^{-1} \log(1 - 0.56) = 0.821\lambda^{-1} \\ u = 0.85 &\implies y = -\lambda^{-1} \log(1 - 0.85) = 1.897\lambda^{-1} \\ u = 0.26 &\implies y = -\lambda^{-1} \log(1 - 0.26) = 0.301\lambda^{-1} \end{aligned}$$

Q2

> Let the random variable X have pdf:
>
> $$f(x \mid \mu) = \begin{cases} \mu^2 x e^{-\mu x} & \text{if } x \in [0, \infty) \\ 0 & \text{otherwise} \end{cases}$$
>
> where $\mu > 0$.
>
> Show that the Exponential distribution can be used as a proposal distribution in a rejection sampler to generate simulations of X. Ensure you state any conditions on λ and μ.

We require $c < \infty$ such that

$$\mu^2 x e^{-\mu x} \le c \lambda e^{-\lambda x} \quad \forall\, x \in [0, \infty)$$

In other words, we require:

$$c = \sup_{x \in [0, \infty)} \frac{\mu^2}{\lambda} x e^{(\lambda - \mu) x}$$

Firstly, we note that at $x = 0$ the expression is zero, and as $x \to +\infty$ the exponential will decay faster than the linear term in x grows *as long as* it has negative power, meaning $\mu > \lambda$.

Next, to find c,

$$\frac{\partial}{\partial x}\left(\frac{\mu^2}{\lambda}xe^{(\lambda-\mu)x}\right) = \frac{\mu^2}{\lambda}\left(e^{(\lambda-\mu)x} + (\lambda-\mu)xe^{(\lambda-\mu)x}\right) \quad \text{(product rule)}$$
$$= \frac{\mu^2}{\lambda}\left(1+(\lambda-\mu)x\right)e^{(\lambda-\mu)x}$$

This derivative is clearly only zero if,

$$1+(\lambda-\mu)x = 0 \implies x = \frac{1}{\mu-\lambda}$$

which will never be undefined since we require $\mu > \lambda$ from above.

Therefore, we have shown $\tilde{f}$ is a suitable proposal in a rejection sampler for f when $\mu > \lambda$ and has bounding constant:

$$c = \frac{\mu^2}{\lambda}\frac{1}{\mu-\lambda}e^{(\lambda-\mu)\frac{1}{\mu-\lambda}}$$
$$= \frac{\mu^2}{\lambda(\mu-\lambda)}e^{-1}$$

Q3

> For any choice of μ, what is the optimal λ to choose as the parameter in the proposal distribution?

We recall that c is the expected number of iterations of the rejection sampling algorithm until acceptance. Therefore, the optimal choice of λ will be the one which leads to the smallest value for c. To find this, we differentiate wrt λ:

$$\frac{\partial}{\partial \lambda}\left[\frac{\mu^2 e^{-1}}{\lambda(\mu-\lambda)}\right] = \frac{\lambda(\mu-\lambda)\times 0 - 1\times\frac{\partial}{\partial\lambda}\left[\lambda(\mu-\lambda)\right]}{\lambda^2(\mu-\lambda)^2} \quad \text{(quotient rule)}$$
$$= \frac{e^{-1}\mu^2(\mu-2\lambda)}{\lambda^2(\mu-\lambda)^2} \quad \text{(product rule)}$$

The above is only zero if $\mu-2\lambda=0$. Note from the question that $\mu>0$, and from solving Q2 that $\mu>\lambda$, so there is no risk that $\mu-\lambda=0$ in the denominator.

Therefore, the optimal choice is $\lambda = \frac{\mu}{2}\ \forall\, \mu > 0$.

Q4

> Show that when the optimal λ is used the expected number of iterations required to produce a single simulation of X is approximately 1.47 for all μ.

We simply substitute the optimal choice of λ back into the expression for c,

$$c = \frac{\mu^2}{\lambda(\mu-\lambda)}e^{-1}$$
$$= \frac{\mu^2}{\frac{\mu}{2}\left(\mu-\frac{\mu}{2}\right)}e^{-1}$$
$$= 4e^{-1} \approx 1.47$$

as required.