# Data Science and Statistical Computing

# Assignment 3 Solutions

## Q1

The following integral is difficult to evaluate analytically:

$$\int_{-1}^{1} \frac{x^2}{\sqrt{x^2+4}}\, dx$$

Express the integral as an expectation with respect to a Uniform distribution and write down the Monte Carlo estimator of this integral given simulations $\{x_1, \dots, x_n\}$ from a Uniform$(-1, 1)$.

$$\begin{aligned} \int_{-1}^{1} \frac{x^2}{\sqrt{x^2+4}}\, dx &= \int_{-1}^{1} \left( \frac{2x^2}{\sqrt{x^2+4}} \right) \underbrace{\frac{1}{2}}_{\text{Uniform(-1,1) pdf}} dx \\ &= \mathbb{E}_X \left[ \frac{2X^2}{\sqrt{X^2+4}} \right] \quad \text{for } X \text{ Uniform(-1,1)} \end{aligned}$$

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} \frac{2x_i^2}{\sqrt{x_i^2+4}}$$

## Q2

The function $f_X(x) = \frac{3}{2}x^2$ is a valid probability density function for a random variable $X \in [-1, 1]$, and $X$ can easily be simulated (we will see how soon!)

Express the integral as an expectation with respect to the random variable $X$ having this pdf, and write down the Monte Carlo estimator of this integral given simulations $\{x_1, \dots, x_n\}$ of $X$.

$$\begin{aligned} \int_{-1}^{1} \frac{x^2}{\sqrt{x^2+4}}\, dx &= \int_{-1}^{1} \left( \frac{2}{3\sqrt{x^2+4}} \right) \underbrace{\frac{3}{2}x^2}_{\text{pdf } f(x)} dx \\ &= \mathbb{E}_X \left[ \frac{2}{3\sqrt{X^2+4}} \right] \quad \text{for } X \text{ following } f(x) \end{aligned}$$

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} \frac{2}{3\sqrt{x_i^2+4}}$$

## Q3

You do a pilot simulation and find the terms in the Monte Carlo estimator for (Q1) have variance $0.0731$, whilst for (Q2) they have variance $8.12 \times 10^{-5}$. Determine how many simulations would be needed to achieve a RMSE of $0.00005$ in each case.

The RMSE is:

$$\frac{\sqrt{\text{Var}(Y)}}{\sqrt{n}}$$

We can use the simulations to also estimate $\text{Var}(Y)$ in each case. Therefore, for the Uniform case, we need:

$$\begin{aligned}\frac{\sqrt{\text{Var}(Y)}}{\sqrt{n}} &= 0.00005 \\ \implies n &= \frac{0.0731}{0.00005^2} \\ &= 29,240,000\end{aligned}$$

That is, we need $29,240,000$ simulations where

$$Y = \frac{2X^2}{\sqrt{X^2+4}}$$

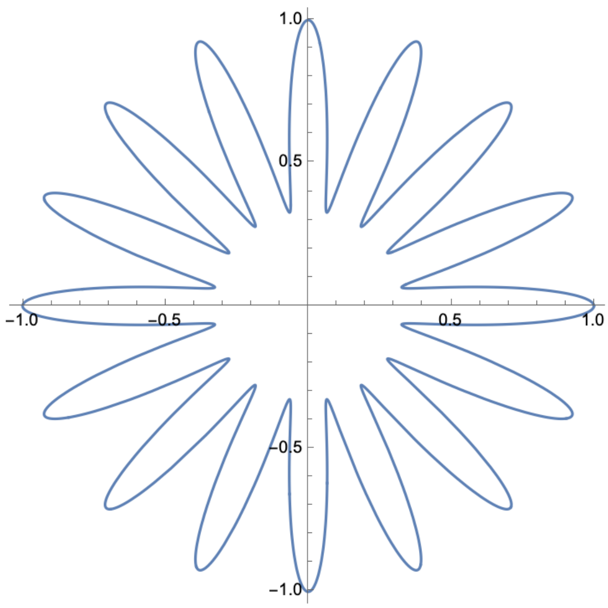
For $f(x)$, we need:

$$\begin{aligned}\frac{\sqrt{\text{Var}(Y)}}{\sqrt{n}} &= 0.00005 \\ \implies n &= \frac{8.12 \times 10^{-5}}{0.00005^2} \\ &= 32,480\end{aligned}$$

Thus, we need $32,480$ simulations where

$$Y = \frac{2}{3\sqrt{X^2+4}}$$

## Q4



The area inside the blue curved "flower" above is defined in polar coordinates as the region,

$$\left\{(r,\theta) : r \leq \frac{2}{3} + \frac{1}{3}\cos(16\theta),\ 0 \leq \theta \leq 2\pi\right\}$$

**Note:** you do not need to do lots of algebra here, think about this particular problem geometrically.

Write R code to estimate the area within the flower based on Uniform simulations on the square $[-1,1]^2$ (just like we estimated $\pi$ in lectures). Run this code to create a Monte Carlo estimate based on $10,000$ simulations from the square and provide a 95% confidence interval for your estimate.

**Hint:** you may find the R function `atan2()` useful, see "Details" on the function help page.

```
n <- 10000
x <- runif(n, -1, 1)
y <- runif(n, -1, 1)

p.hat <- mean(sqrt(x^2+y^2) <= 2/3 + 1/3*cos(16*atan2(y,x)))
p.std.err <- sqrt((p.hat*(1-p.hat))/n)

A.hat <- 4*p.hat
A.ci <- 4*(p.hat + c(-1,1)*1.96*p.std.err)

A.hat
```

```
[1] 1.5736
```

```
A.ci
```

```
[1] 1.535301 1.611899
```

It was not asked in the question, but notice the size of the CI here is 0.0765975.

## Q5

Notice that the flower is contained entirely within the circle of radius 1 centred at $(0,0)$. It is probably more natural given the polar form to produce your Monte Carlo estimate using uniform simulations in the circle.

Write R code which simulates a uniformly random point in the circle by simulating $\theta \sim$ Unif$(0,2\pi)$ and $r^2 \sim$ Unif$(0,1)$, then use this code to create a Monte Carlo estimate based on $10,000$ simulations from the circle and provide a 95% confidence interval for your estimate.

**Hint:** simulate a Uniform on $(0,1)$ and take the square root to simulate $r$ above.

**Aside:** it may initially seem counter-intuitive, but if you simulate $r$ as just a straight Uniform, then you do *not* end up with points uniformly distributed over the circle because the density of points near the origin will be too high. The square root transformation ensures uniformity over the circle.

```
n <- 10000
theta <- runif(n, 0, 2*pi)
r <- sqrt(runif(n, 0, 1))

p.hat <- mean(r <= 2/3 + 1/3*cos(16*theta))
p.std.err <- sqrt((p.hat*(1-p.hat))/n)
```

```r
A.hat <- pi*p.hat
A.ci <- pi*(p.hat + c(-1,1)*1.96*p.std.err)

A.hat
```

```
[1] 1.547234
```

```r
A.ci
```

```
[1] 1.516450 1.578019
```

It was not asked in the question, but notice that the size of the CI here is 0.0615683, which should be smaller than in Q4 for the same simulation size $n$.