Data Science and Statistical Computing

Assignment 3

Due Monday 25th November 2024 at noon in Gradescope

Part A

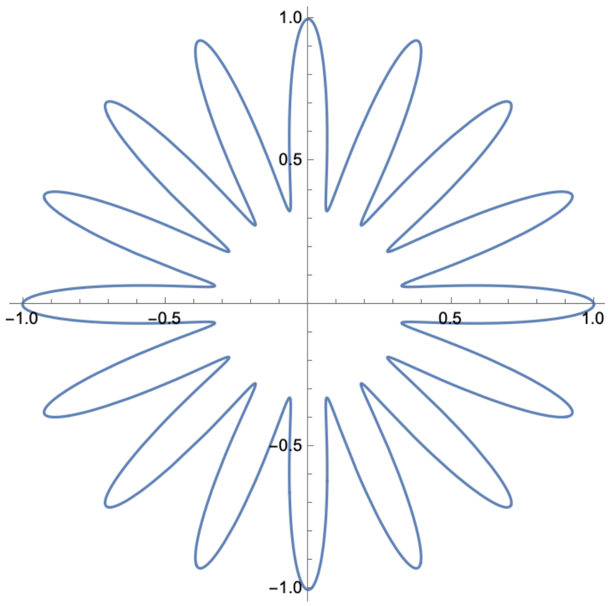
The following integral is difficult to evaluate analytically:

$$\int_{-1}^{1} \frac{x^2}{\sqrt{x^2+4}}\,dx$$

1. Express the integral as an expectation with respect to a Uniform distribution and write down the Monte Carlo estimator of this integral given simulations $\{x_1, \dots, x_n\}$ from a Uniform$(-1, 1)$.
2. The function $f_X(x) = \frac{3}{2}x^2$ is a valid probability density function for a random variable $X \in [-1, 1]$, and X can easily be simulated (we will see how soon!)

   Express the integral as an expectation with respect to the random variable X having this pdf, and write down the Monte Carlo estimator of this integral given simulations $\{x_1, \dots, x_n\}$ of X.
3. You do a pilot simulation and find the terms in the Monte Carlo estimator for (Q1) have variance 0.0731, whilst for (Q2) they have variance 8.12×10^{-5}. Determine how many simulations would be needed to achieve a RMSE of 0.00005 in each case.

Part B



The area inside the blue curved "flower" above is defined in polar coordinates as the region,

$$\left\{(r,\theta) : r \le \frac{2}{3} + \frac{1}{3}\cos(16\theta),\ 0 \le \theta \le 2\pi\right\}$$

Note: you do not need to do lots of algebra here, think about this particular problem geometrically.

4. Write R code to estimate the area within the flower based on Uniform simulations on the square $[-1, 1]^2$ (just like we estimated π in lectures). Run this code to create a Monte Carlo estimate based on $10,000$ simulations from the square and provide a 95% confidence interval for your estimate.

   Hint: you may find the R function `atan2()` useful, see "Details" on the function help page.

5. Notice that the flower is contained entirely within the circle of radius 1 centred at $(0, 0)$. It is probably more natural given the polar form to produce your Monte Carlo estimate using uniform simulations in the circle.

   Write R code which simulates a uniformly random point in the circle by simulating $\theta \sim \text{Unif}(0, 2\pi)$ and $r^2 \sim \text{Unif}(0, 1)$, then use this code to create a Monte Carlo estimate based on $10,000$ simulations from the circle and provide a 95% confidence interval for your estimate.

   Hint: simulate a Uniform on $(0, 1)$ and take the square root to simulate r above.

   Aside: it may initially seem counter-intuitive, but if you simulate r as just a straight Uniform, then you do *not* end up with points uniformly distributed over the circle because the density of points near the origin will be too high. The square root transformation ensures uniformity over the circle.