Networks with Repairable Redundant Subsystems

Faster Inference for Phase-type Distributions

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1. The Problem

Phase-type (PHT) distributions are a natural choice for modelling the multiple phases of failure and repair which a redundant subsystem may go through before ultimately becoming unavailable.

Bladt et al. (2003) present a scheme for Bayesian inference on general PHT distributions. There are some key areas where there is scope to extend this work in a reliability context:

i) the need to account for censoring and other common situations in reliability;

ii) the need for parameter constraints;

iii) the sampling scheme is intractably slow for the PHT distributions commonly encountered in reliability.

3. Bladt et al. (2003) Algorithm

Full stochastic process to absorption observed $\implies \exists$ conjugate priors $\boldsymbol{\pi} \sim \text{Dir}$; $S_{ij}, s_i \sim \text{Gam}$

Bladt et al. (2003) proposed a Metropolis-Hastings (MH) within Gibbs sampler for the unobserved process case.

- MH proposal is draw from:

  $$p(\text{path} \cdot \,|\, \boldsymbol{\pi}, \mathbf{S}, Y \geq y_i)$$

  by rejection sampling.

  Acceptance ratio $\implies$ last sample from

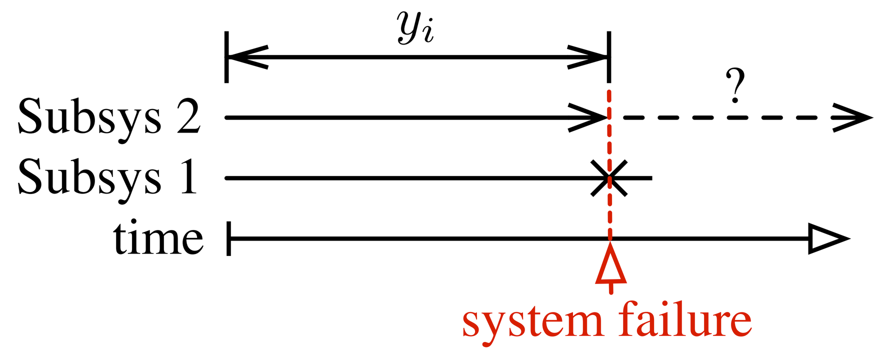
  $$p(\text{path} \cdot \,|\, \boldsymbol{\pi}, \mathbf{S}, Y = y_i)$$

  after truncating to y_i.
- sample from unobserved process in MH step gives conjugacy for Gibbs step.

$$p(\boldsymbol{\pi}, \mathbf{S} \,|\, \text{paths } \cdot, \mathbf{y})$$
$$p(\text{paths } \cdot \,|\, \boldsymbol{\pi}, \mathbf{S}, \mathbf{y})$$

4. Censoring/Constraints

Censoring: arises through competing risks:



Elegantly dealt with by performing just rejection sampling part of MH step.

Parameter Constraints: We have shown that, with possible prior parameter restrictions, Gibbs step conjugacy can be maintained when imposing constraints such as:

$$\mathcal{C}_1: \quad S_{12} = S_{13} = s_2 = s_3 = \lambda_f$$
$$\mathcal{C}_2: \quad S_{23} = 0$$

This is desirable for applications and also reduces the dimension of the parameter space.

2. PHT Distributions

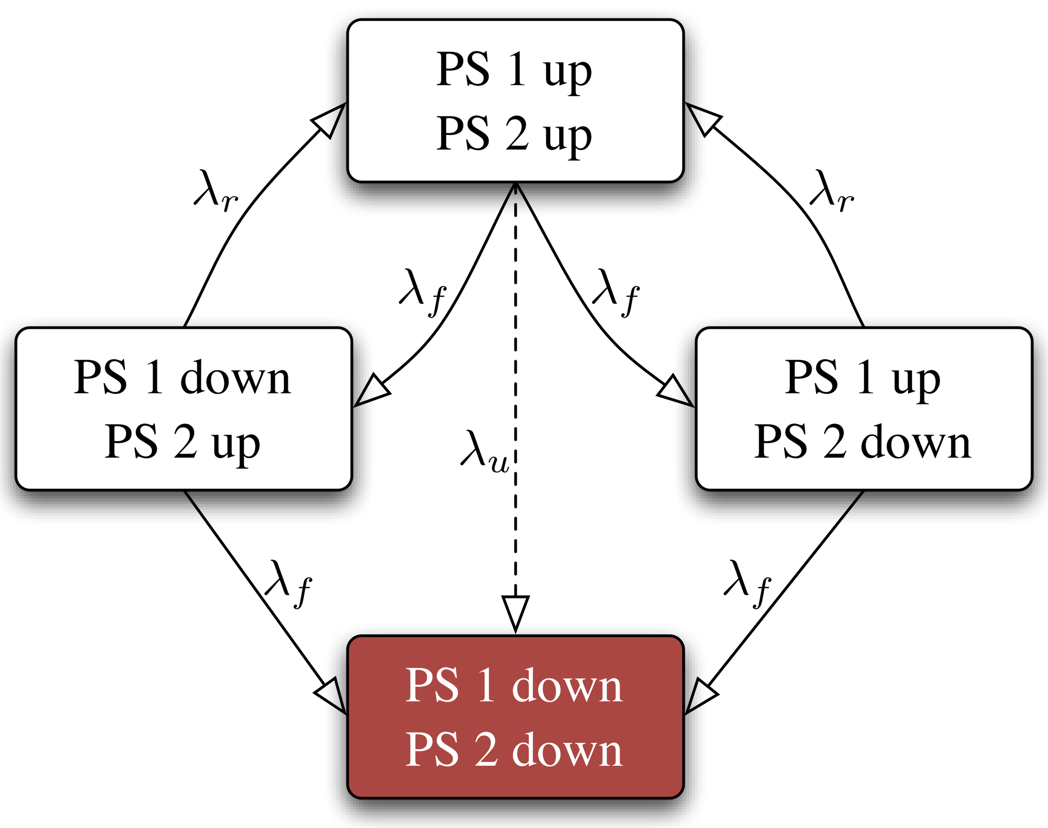
Consider a Continuous-Time Markov Chain (CTMC) with an absorbing state. Without loss of generality, let the CTMC generator be written:

$$\mathbf{T} = \begin{pmatrix} \mathbf{S} & \mathbf{s} \\ \mathbf{0}^{\text{T}} & 0 \end{pmatrix}$$

Then, if X is the random variable denoting time to entering the absorbing state, $X \sim \text{PHT}(\boldsymbol{\pi}, \mathbf{T})$ and

$$\begin{aligned} F_X(x) &= 1 - \boldsymbol{\pi}^{\text{T}} \exp\{x\mathbf{S}\}\mathbf{e}, \quad \mathbf{e} = (1, \ldots, 1)^{\text{T}} \\ f_X(x) &= \boldsymbol{\pi}^{\text{T}} \exp\{x\mathbf{S}\}\mathbf{s} \end{aligned}$$

Simplest Example: Consider a dual redundant hot-swappable power supply (PS) subsystem.



λ_r : Repair rate; λ_f : Failure Rate; λ_u : Uncovered Failure Rate (ignore for simple case).

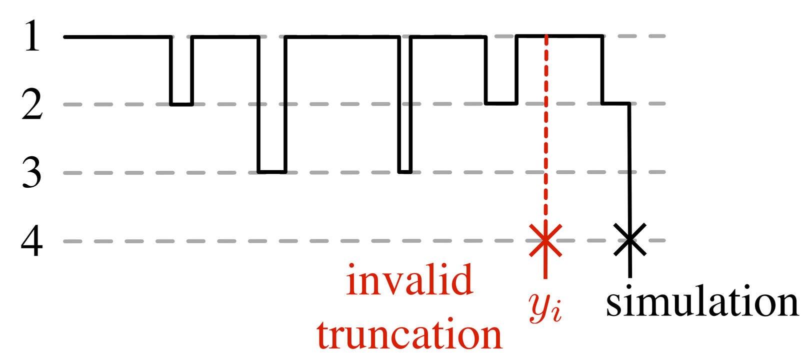
$$\implies \mathbf{T} = \begin{pmatrix} -2\lambda_{\text{f}} & \lambda_{\text{f}} & \lambda_{\text{f}} & 0 \\ \lambda_{\text{r}} & -\lambda_{\text{r}} - \lambda_{\text{f}} & 0 & \lambda_{\text{f}} \\ \lambda_{\text{r}} & 0 & -\lambda_{\text{r}} - \lambda_{\text{f}} & \lambda_{\text{f}} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

5. Computational Tractability

The most significant advance is computational. Consider the simple example presented in box 2 (with $\lambda_{\text{r}} \approx 30^{-1}$ and $\lambda_{\text{f}} \approx 100000^{-1}$, say).

a) Even small moves on the Gibbs step can result in samples of $\boldsymbol{\pi}$ and $\mathbf{S}$ such that observations y_i are so extreme in the right PHT tail as to stall the rejection sampling;

b) Furthermore, with $T_{14} = 0$ there are significant issues with 'invalid' MH proposals: when truncating to time y_i, if the CTMC is in state 1 an invalid absorbing move $1 \to 4$ is inserted.



We propose two advances to remedy this.

a) Direct Conditional Sampling: Rather than rejection sampling, which is susceptible to stalling, it is more desirable to sample $p(\text{path} \cdot \,|\, \boldsymbol{\pi}, \mathbf{S}, Y \geq y_i)$ directly. This requires the ability to sample from the conditional sojourn time density:

$$p(\delta = \Delta \,|\, \boldsymbol{\pi}, \mathbf{S}, j(t), t, Y_i \geq y_i)$$

where t is the current time and $j(t)$ the current state in the CTMC path being sampled.

We have shown this can be calculated as:

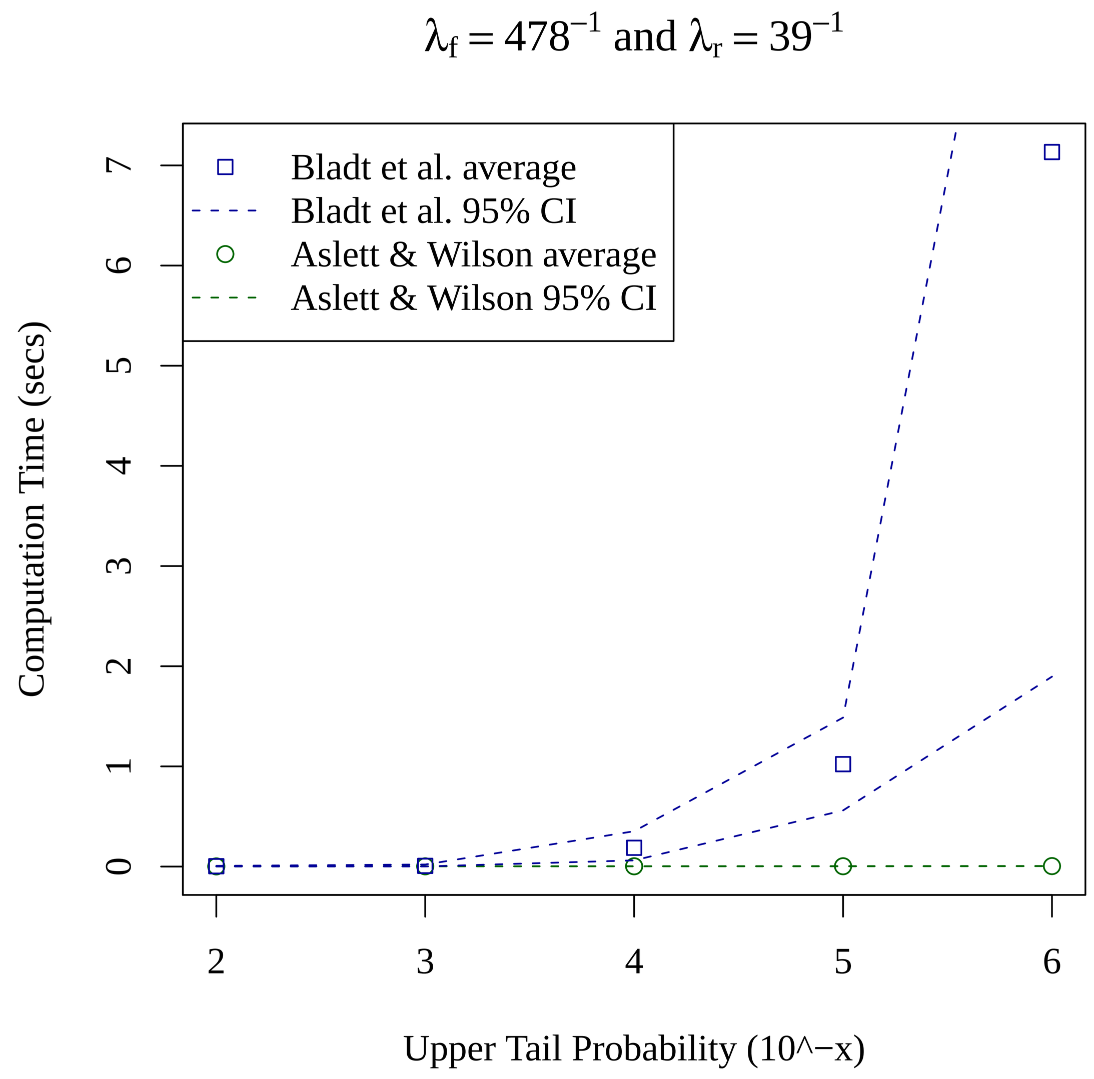
$$= \frac{\left(\sum_{k=1}^{n} p(j(t+\Delta) = k \,|\, j(t), \mathbf{T}, t) p(Y_i \geq y_i - t - \Delta \,|\, \mathbf{T}) \right) p(\delta = \Delta \,|\, j(t), \mathbf{T}, t)}{p(Y_i \geq y_i \,|\, j(t), \mathbf{T}, t)}$$

This is nearly log-linear, though not log-concave. Adaptive Rejection Metropolis Sampling (ARMS) has proven highly efficient.

b) Reverse Simulation: For highly reliable systems, the starting state (full operation) is the most common. Thus, by sampling in reverse from y_i and truncating at 0 the commonality of state 1 becomes a major advantage and 'invalid' proposals are rare.

This requires detailed balance to be satisfied. Also, absorbing CTMCs don't necessarily reach stationarity, so selection of starting state must be made from the quasi-stationary distribution.

Speed Comparison:



References

Bladt, M., Gonzalez, A. & Lauritzen, S. L. (2003), 'The estimation of phase-type related functionals using Markov chain Monte Carlo methods', *Scandinavian Journal of Statistics* **2003**(4), 280–300.

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