Inference on Phase-type Models via MCMC

with application to repairable redundant systems

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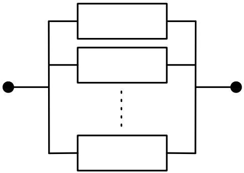
Reliability Theory

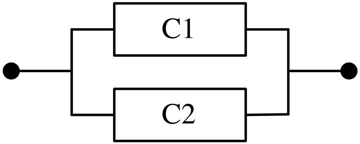
- Simplest situation: single component modelled with lifetime distribution.
- Redundant collection of components: e.g. components in parallel.



  Often assume no repair. Once component goes down, it stays down.

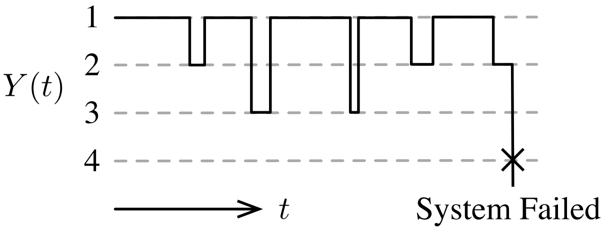
- Repairable redundant collection of components $\implies$ now need to consider a general stochastic process.

Toy Example : Redundant Repairable Components



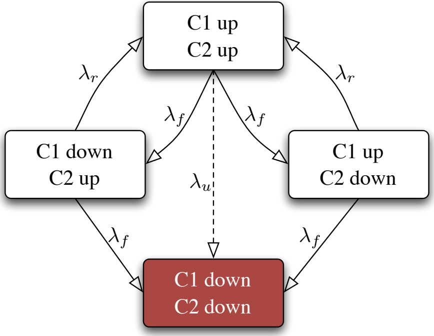
State	Meaning
1	both C1 and C2 work
2	C1 failed, C2 working
3	C1 working, C2 failed
4	system failed

$\therefore$ a general stochastic process, e.g.



Continuous-time Markov Chain Model

State	Meaning
1	both C1 and C2 work
2	C1 failed, C2 working
3	C1 working, C2 failed
4	system failed



$$\implies \boldsymbol{\pi} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{T} = \begin{pmatrix} -2\lambda_\mathrm{f} & \lambda_\mathrm{f} & \lambda_\mathrm{f} & 0 \\ \lambda_\mathrm{r} & -\lambda_\mathrm{r} - \lambda_\mathrm{f} & 0 & \lambda_\mathrm{f} \\ \lambda_\mathrm{r} & 0 & -\lambda_\mathrm{r} - \lambda_\mathrm{f} & \lambda_\mathrm{f} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Definition of Phase-type Distributions

An absorbing continuous time Markov chain is one in which there is a state that, once entered, is never left. That is, the $n+1$ state intensity matrix can be written:

$$\mathbf{T} = \begin{pmatrix} \mathbf{S} & \mathbf{s} \\ \mathbf{0} & 0 \end{pmatrix}$$

where $\mathbf{S}$ is $n \times n$, $\mathbf{s}$ is $n \times 1$ and $\mathbf{0}$ is $1 \times n$, with

$$\mathbf{s} = -\mathbf{S}\mathbf{e}$$

Then, a *Phase-type distribution* (PHT) is defined to be the distribution of the time to entering the absorbing state.

$$Y \sim \text{PHT}(\boldsymbol{\pi}, \mathbf{S}) \implies \begin{cases} F_Y(y) = 1 - \boldsymbol{\pi}^{\text{T}} \exp\{y\mathbf{S}\}\mathbf{e} \\ f_Y(y) = \boldsymbol{\pi}^{\text{T}} \exp\{y\mathbf{S}\}\mathbf{s} \end{cases}$$

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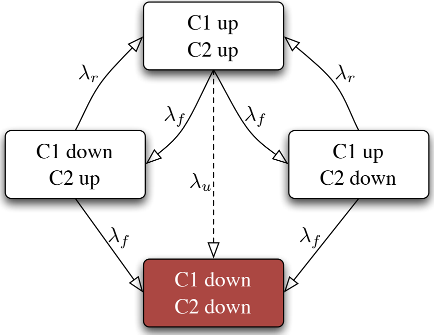
Relating to the Toy Example

State	Meaning
1	both PS working
2	1 failed, 2 working
3	1 working, 2 failed
4	subsystem failed



$$\implies \mathbf{T} = \left(\begin{array}{ccc|c} -2\lambda_f & \lambda_f & \lambda_f & 0 \\ \lambda_r & -\lambda_r - \lambda_f & 0 & \lambda_f \\ \lambda_r & 0 & -\lambda_r - \lambda_f & \lambda_f \\ \hline 0 & 0 & 0 & 0 \end{array}\right)$$

(Upper-left block labelled $\mathbf{S}$, upper-right column labelled $\mathbf{s}$.)

$$f_Y(y) = \boldsymbol{\pi}^{\mathrm{T}} \exp\{y\mathbf{S}\}\mathbf{s} \qquad F_Y(y) = 1 - \boldsymbol{\pi}^{\mathrm{T}} \exp\{y\mathbf{S}\}\mathbf{e}$$

Inferential Setting

Cano & Rios (2006) provide conjugate posterior calculations in the context of analysing repairable systems when stochastic process leading to absorption is observed.

Data

For each system failure time, one has:

- Starting state
- Length of time in each state
- Number of transitions between each state
- Ultimate system failure time

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Reduced information scenario $\implies$ Bladt et al. (2003) provide a Bayesian MCMC algorithm, or Asmussen et al. (1996) provide a frequentist EM algorithm.

Slide for Statisticians!

Strategy is a top-level Gibbs step which achieves the goal of simulating from

$$p(\boldsymbol{\pi}, \mathbf{S} \,|\, \mathbf{y})$$

by sampling from

$$p(\boldsymbol{\pi}, \mathbf{S}, \text{paths } \cdot \,|\, \mathbf{y})$$

through the iterative process

$$p(\boldsymbol{\pi}, \mathbf{S} \,|\, \text{paths } \cdot, \mathbf{y})$$

$$p(\text{paths } \cdot \,|\, \boldsymbol{\pi}, \mathbf{S}, \mathbf{y})$$

where $p(\text{paths } \cdot \,|\, \boldsymbol{\pi}, \mathbf{S}, \mathbf{y})$ is achieved by a rejection sampling within Metropolis-Hastings algorithm.

High-level Description of Bladt et al.

The following are key points to note about the MCMC scheme:

- fully dense rate matrix with separate parameters, e.g.

$$\mathbf{T} = \begin{pmatrix} \cdot & S_{12} & S_{13} & s_1 \\ S_{21} & \cdot & S_{23} & s_2 \\ S_{31} & S_{32} & \cdot & s_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- no censored data
- slow computational speed in some common scenarios
- focused on 'distribution fitting'

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  → **we extend to allow structure to be imposed**
- no censored data
  → **we accommodate censoring**
- slow computational speed in some common scenarios
  → **we provide novel sampling scheme**
- focused on 'distribution fitting'
  → **all together shifts focus to stochastic modelling**

Statistical -vs- Stochastic

In other words, we adapt the MCMC algorithm to be fit for performing inference when PHTs used for stochastic rather than statistical modelling.

Stochastic Model ⟶ Aslett & Wilson

"Stochastic models seek to represent an underlying physical phenomenon of interest, albeit often in a highly idealised way, and have parameters that are physically interpretable." — Isham

Statistical Model ⟶ Bladt et al

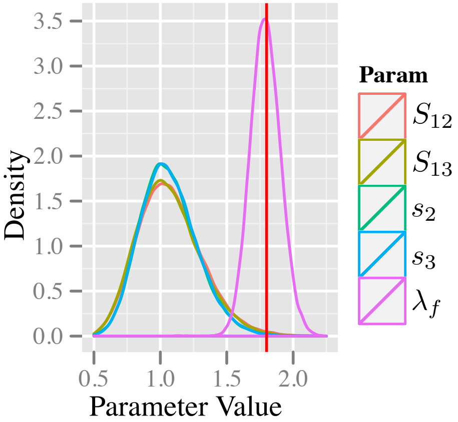
"In contrast, statistical models are descriptive, and represent the statistical properties of data and their dependence on covariates, without aiming to encapsulate the physical mechanisms involved." — Isham

Toy Example Results

100 uncensored observations simulated from PHT with

$$\mathbf{S} = \begin{pmatrix} -3.6 & 1.8 & 1.8 \\ 9.5 & -11.3 & 0 \\ 9.5 & 0 & -11.3 \end{pmatrix}$$

$$\implies \lambda_f = 1.8,\ \lambda_r = 9.5$$



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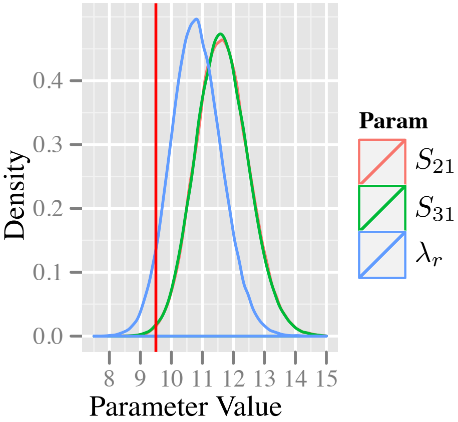

Reliability less sensitive to λ_r (Daneshkhah & Bedford 2008)

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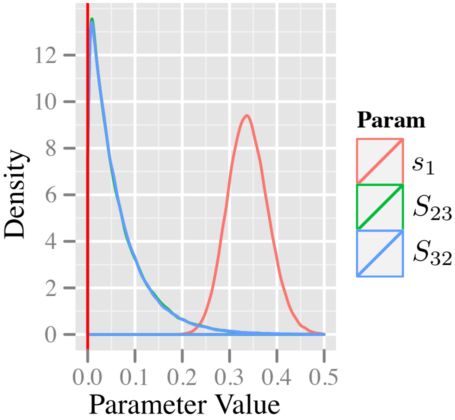
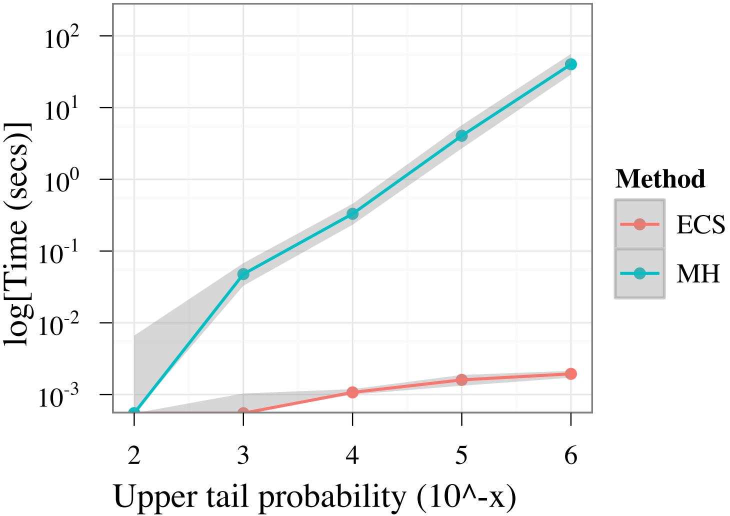


'Tail Depth' Performance Improvement



Overall Performance Improvement

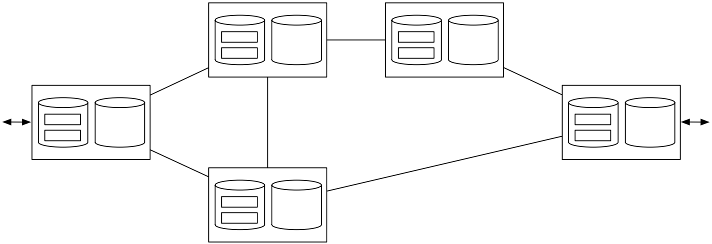
This shows the new method keeping pace in 'nice' problems and significantly outperforming otherwise.

$\mathbf{T} = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$			$\mathbf{T} = \begin{pmatrix} -2 & 0.01 & 1.99 & 0 \\ 1 & -300 & 0 & 299 \\ 299 & 0 & -300 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	
No problems i-iii			All problems i-iii	
	MH	ECS	MH	ECS
$\bar{t}$	1.6 µs	7.2 µs	10.2 hours	0.016 secs
s_t	104 µs	19 µs	9.4 hours	0.015 secs

2,300,000 × faster on average in hard problem

Future Work

- Further computational work
  matrix exp/functional approximation/autocorrelation
- Study networks of repairable redundant systems modelled by Phase-types using the extended MCMC methodology



Hierarchical Bayesian inference.

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Asmussen, S., Nerman, O. & Olsson, M. (1996), 'Fitting phase-type distributions via the EM algorithm', *Scandinavian Journal of Statistics* **23**(4), 419–441.

Bladt, M., Gonzalez, A. & Lauritzen, S. L. (2003), 'The estimation of phase-type related functionals using Markov chain Monte Carlo methods', *Scandinavian Journal of Statistics* **2003**(4), 280–300.

Cano, J. & Rios, D. (2006), 'Reliability forecasting in complex hardware/software systems', *Proceedings of the First International Conference on Availability, Reliability and Security (ARES'06)* .

Daneshkhah, A. & Bedford, T. (2008), 'Sensitivity analysis of a reliability system using gaussian processes', *Advances in mathematical modeling for reliability* pp. 46–62.